ISSN: 2277-5536 (Print); 2277-5641 (Online)

GAMMA FUNCTION, BETA FUNCTIONS AND ITS APPLICATIONS IN THE DEVELOPMENTS OF FRACTIONAL DERIVATIVE

Divate B. B.* and Panchal S. K.**
*Department of Mathematics, H.P.T. Arts and R.Y.K. Science College, Nashik 422005, India.
E-Mail: rykmathsbbd@gmail.com
**Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-431004, India.
E-mail: drskpanchal@gmail.com

ABSTRACT
The aim of this paper is to study gamma and beta functions of complex variable. Further, we prove some properties of gamma and beta functions of complex variables, which are useful to define the fractional integral and fractional derivative. Also, we prove some elementary properties of fractional derivatives.

KEY WORDS: Beta Functions, Gamma, Fractional Derivative.

INTRODUCTION

Fractional calculus is a field of mathematical study that grows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Most authors on this topic will cite a particular date as the birthday of so called "fractional calculus". In letter dated September, 30, 1965 L'Hospital wrote to Leibniz asking him about a particular notation he had used in his publications for the n^{th}derivative of linear function$f(x) = x$, $\frac{d^n x}{dx^n}$. L'Hospital posed the question to Leibniz, what would be the result if $n = 1/2$? From this question fractional calculus was born. Fourier, Euler and Laplace worked on fractional calculus and the mathematical consequences.

Many researchers found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the word of fractional calculus are the Riemann-Liouville and *Gru¨nwald*-Letnikov definitions (Oldham, 1974; Podlubny, 1999). Most of the mathematical theory applicable to study of fractional calculus is developed in nineteenth century (Bride and Roach, 1985; Hilfer, 2000; Miller and Ross, 1993; Nishimoto, 1990). Understanding of definitions and use of fractional calculus requires the deep knowledge of Gamma function, Beta function, Laplace transform and Mittag-Leffler function.

We organize this paper as follows: In section 2, we define gamma and beta functions for complex variable and study their properties. The section 3 is devoted for the fractional derivative of some functions. In the last section we prove some elementary properties of fractional derivatives.

2 . GAMMA AND BETA FUNCTIONS AND THEIR PROPERTIES:

Gamma Function: This is the one of the basic function of fractional calculus [6, 7]. The gamma function for complex variable z with $Re.(z) > 0$ is denoted by $\Gamma(z)$ and is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \qquad (2.1)$$

which converges in right half plane of the complex plane *Re*(*z*) > 0. Equation (2.1) can be written as

$\Gamma(x+iy) = \int_0^\infty e^{-t} t^{x-1+iy} dt$, where $z = x + iy$

$$= \int_0^\infty e^{-t} t^{x-1} t^{iy} dt$$

$$= \int_0^\infty e^{-t} t^{x-1} e^{iy(\log t)} dt$$

$$= \int_0^\infty e^{-t} t^{x-1} [\cos(y\log t) + i\sin(y\log t)] dt \qquad (2.2)$$

Since $|\cos(y\log t) + i\sin(y\log t)| = 1; \forall t$ and hence convergence of integral (2.2) at infinity is provided by e^{-t}, and for the convergence at *t* = 0 we must have $x = Re.(z) > 1$.

Beta Function: It is more convenient to use Beta function of a certain combination of values of gamma function. The beta function is usually defined by

$$\beta(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt, \quad (Re(z) > 0, Re(w) > 0) \qquad (2.3)$$

Properties:

$\beta(z,w) = \beta(w,z)$

Proof: Now consider the integral,

$$h_{z,w}(t) = \int_0^t t^{z-1}(1-t)^{w-1} dt \qquad (2.4)$$

where $h_{z,w}(t)$ is a convolution of the functions t^{z-1} and t^{w-1} and

$h_{z,w}(1) = \beta(z,w)$

Applying the Laplace transform of convolution on both sides of the equation (2.4), we have

$$H_{z,w}(s) = \frac{\Gamma(z)\Gamma(w)}{s^z s^w} = \frac{\Gamma(z)\Gamma(w)}{s^{z+w}} \qquad (2.5)$$

where $H_{z,w}(s)$ is the Laplace transform of function $h_{z,w}(t)$.
The inverse Laplace transform of equation (2.5) is given by

$$h_{z,w}(s) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} t^{z+w-1} \qquad (2.6)$$

By taking $\leqslant = 1$, we get

$$\beta(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \frac{\Gamma(z)\Gamma(w)}{\Gamma(w+z)} = \beta(w,z) \qquad (2.8)$$

The definition of beta function (2.3) is valid only for $Re(z) > 0; Re(w) > 0$. The relation (2.7) provides the analytical continuation of beta function for entire complex plane.

$\Gamma(z+1) = z\Gamma(z)$

Proof:

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z dt$$
$$= [e^{-t}t^z]_0^\infty - \int_0^\infty \left(\int e^{-t} \frac{d}{dt} t^z dt\right) dt$$
$$= -\int_0^\infty e^{-t} z\, t^z dt$$
$$= z\int_0^\infty e^{-t} t^z dt$$
$$= z\Gamma(z)$$

Note: $\Gamma(1) = 1$

$\Gamma(2) = 1\Gamma(1) = 1!$

$\Gamma(3) = 2\Gamma(2) = 2!$

$\vdots$

$\Gamma(n+1) = n\Gamma(n) = n!$

$\Gamma(z)$ has simple poles at the points $z = 0, -1, -2, \dots$.

Proof: We have by the definition

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$
$$= \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt \qquad (2.9)$$

Where

$$\int_0^1 e^{-t} t^{z-1} dt = \int_0^1 \sum_{k=0}^\infty \frac{(-t)^k}{k!} t^{z-1} dt$$
$$= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_0^1 t^{k+z-1} dt \qquad (2.10)$$
$$= \sum_{k=0}^\infty \frac{-t^k}{k!} \frac{t^{k+z}}{(k+z)} \qquad (2.11)$$

And, second integral in equation (2.9) is

$$\varphi(z) = \int_1^\infty e^{-t} t^{z-1} dt$$
$$= \int_1^\infty e^{-t} e^{(z-1)logt} dt$$
$$= \int_1^\infty e^{[(z-1)logt - t]} dt \qquad (2.12)$$

Since $e^{[(z-1)logt-t]}$ is continuous function of z and t for $t \geq 1$ and $logt \geq 0$ for $t \geq 1$ implies $e^{[(z-1)logt-t]}$ is an entire function of z.

Let us consider an arbitrary bounded closed domain *D* in the complex plane and let $x0 = \max_{z\in D} Re(z)$.

Then we have

$$|e^{-t}t^{z-1}| = |e^{[(z-1)logt-t]}|$$
$$= |e^{[(x+iy-1)logt-t]}|$$
$$= |e^{[(x-1)logt-t]} e^{iylogt}|$$

$$= |e^{[(x-1)\log t - t]}|$$
$$\leq |e^{[(x_0-1)\log t - t]}|$$
$$= e^{[(x_0-1)\log t]} e^{-t}$$
$$= e^{-t} t^{x_0-1}$$

Therefore, integral in equation (2.12) converges uniformly in *D* and therefore $\varphi(z)$ is regular function in *D* and hence differentiation under integral in (2.12) is allowed. Since domain *D* has been chosen arbitrarily $\varphi(z)$ has above property in the whole complex plane. Therefore $\varphi(z)$ is an entire function allowing differentiation under the integral

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \int_0^{\infty} e^{-z} t^{z-1} dt$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \varphi(z) \qquad (2.13)$$

And indeed, $\Gamma(z)$ has simple pole at $z = 0, -1, -2, \ldots$

(iv)

$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z,}, \ (0 < Re(z) < 1), z \neq 0, \pm 1, \pm 2, \ldots$

$\frac{\Gamma(z)\Gamma(1-z)}{\Gamma(z+(1-z))} = \beta(z, 1-z)$

$$\Gamma(z)\Gamma(1-z) = \int_0^1 t^{z-1}(1-t)^{1-z-1} dt$$
$$= \int_0^1 t^{z-1} \frac{1}{(1-t)^z} dt$$
$$= \int_0^1 t^{z-1} \frac{1}{(1-t)^z} dt$$
$$= \int_0^1 \left[\frac{t}{(1-t)}\right]^{z-1} \frac{dt}{(1-t)} \qquad (2.14)$$

where the integral converges if $0 < Re(z) < 1$.

By substitution $\tau = \frac{t}{1-t}$, we obtain

$$\Gamma(z)\Gamma(1-z) = \int_0^1 \frac{\tau^{z-1}}{(1+\tau)} dt \qquad (2.15)$$

Now consider the integral $\int_\ell f(s)\,ds$ where $f(s) = \frac{s^{z-1}}{1+s}$

Where ℓ is contour formed by two concentric circles of outer radius $R > 1$ and inner radius ϵ drawn in [7] (page no. 9).

The complex plane is cut along the real positive semiaxis. The function $f(\tau)$ has simple pole at $s = e^{i\pi} = -1$. Therefore, for $R > 1$, we have

$$\int_\ell f(s)ds = 2\pi i[Res f(s)]_{s=-1}$$
$$= 2\pi i \lim_{s\to -1}(s+1)f(s)$$
$$= 2\pi i \lim_{s\to e^{i\pi}} s^{z-1}$$
$$= 2\pi i (e^{i\pi})^{z-1}$$
$$= 2\pi i e^{i\pi z} e^{-i\pi}$$
$$= -2\pi i e^{i\pi z} \qquad (2.17)$$

The integrals along the circumference $|s| = \epsilon$ and $|s| = R$ vanish as $\epsilon \to 0$ and $R \to \infty$. Also, the integral along the upper cut edge differs from the integral along the upper cut edge by the factor $e^{-2\pi z}$.

Therefore, as $\epsilon \to 0$ and $R \to \infty$, we have

$$\int_\ell f(s)\,ds = 2\pi i[Res f(s)]_{s=e^{\pi i}}$$
$$= -2\pi i e^{i\pi z}$$
$$= \Gamma(z)\Gamma(1-z)(1-e^{2\pi z})$$

Therefore,

$$\Gamma(z)\Gamma(1-z) = \frac{2\pi i e^{i\pi z}}{e^{2\pi z} - 1}$$
$$= \frac{\pi}{\sin\pi z}, \ (0 < Re(z) < 1).$$

Moreover, if for integer $m, m < Re(z) < m+1$ then $z = \alpha + m$, where $0 < Re(\alpha) < 1$

$$\Gamma(z)\Gamma(1-z) = \Gamma(\alpha+m)\Gamma(1-(\alpha+m))$$
$$= m!\,\Gamma(\alpha) \frac{(-1)^m}{m!} \Gamma(1-\alpha)$$
$$= (-1)^m \Gamma(\alpha)\Gamma(1-\alpha)$$

$$= \frac{(-1)^m \pi}{\sin \pi\alpha}$$
$$= \frac{\pi}{(-1)^m \sin \pi\alpha}$$
$$= \frac{\pi}{\cos \pi m \sin \pi\alpha}$$
$$= \frac{\pi}{\sin(\pi\alpha + \pi m)}$$
$$= \frac{\pi}{\sin(\pi(\alpha + m))}$$
$$= \frac{\pi}{\sin \pi z}, \ (m < Re(z) < m + 1)$$

Thus,

$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, for $z \neq 0, \pm 1, \pm 2, \ldots$

Taking $z = \frac{1}{2}$, we get $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \frac{\pi}{\sin\frac{\pi}{2}} = \pi$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$
$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$
$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\frac{1}{2}\sqrt{\pi}$$
$$\vdots$$
$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

Again,

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = \sqrt{\pi}\, 2^{2z-1}\Gamma(2z), \ \ 2z \neq 0, -1, -2, \ldots$$

3. FRACTIONAL DERIVATIVE ($Gr\ddot{u}nwald$ –Letnikov Fractional derivative)

We know that $f(x)$ is differentiable function of x if

$D^1 f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, exist

$$D^2 f(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h},$$
$$= \lim_{h\to 0}\frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$$
$$= \lim_{h\to 0} h^{-2}\sum_{m=0}^{2}(-1)^m\binom{2}{m} f(x + (2-m)h), \text{ exist}$$

In general, n^{th} derivative of $f(x)$ is

$$D^n f(x) = \lim_{h\to 0} h^{-n}\sum_{m=0}^{n}(-1)^m\binom{n}{m} f(x + (n-m)h), \text{ exist}$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}$ (3.1)

is valid for nonnegative values.

Now, binomial formula is

$(a+b)^n = \sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m$, for any $n \in N$.

This is generalized to any complex number α as

$$(a+b)^\alpha = \sum_{m=0}^{n}\frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)}a^{\alpha-n}b^n$$

which is convergent if $|b| < a$.

There are some desirable properties that could be required to the fractional derivative

(i) Existence and continuity for n-times derivable function for any n whose modulus is equal or less than m.

(ii) For $n = 0$ the result should be the function itself; for $n > 0$, integer it should be equal to ordinary derivative and for integer $n < 0$ it should be equal to ordinary integration - regardless the integration constant.

(iii) Iteration gives,

$$D^{a+a} f(x) = D^a D^a f(x)$$

D^a is linear operator that is

$$D^{a}\left(af(x) + bg(x)\right) = a\,D^a f(x) + b\,D^a g(x)$$

Allowing Taylor's expansion in other way.

It's characteristics property should be preserved for exponential function,

$$D^a e^x = e^x$$

Exponentials:

$$D^a e^{ax} = \lim_{h\to 0} h^{-a}\sum_{n=0}^{a}(-1)^n\binom{a}{n} e^{a[x+(a-n)h]}$$

$$= e^{ax}\lim_{h\to 0} h^{-\acute{a}}\sum_{n=0}^{\acute{a}}(-1)^n\binom{\acute{a}}{n}(e^{ah})^{\acute{a}-n}$$
$$= e^{ax}\lim_{h\to 0} h^{-\acute{a}}(e^{ah}-1)^{\acute{a}}$$
$$= e^{ax}\lim_{h\to 0}\left[\frac{(e^{ah}-1)}{ah}a\right]^{\acute{a}}$$
$$= a^{\acute{a}}e^{ax}$$

Now,

$$D^{\acute{a}}cosx + i\,D^{\acute{a}}sinx = D^{\acute{a}}[cosx + isinx]$$
$$= D^{\acute{a}}e^{ix}$$
$$= i^{\acute{a}}e^{ix}$$
$$= \left(e^{\frac{\eth i}{2}}\right)^{\acute{a}} e^{ix}$$
$$= e^{i(x+\acute{a}\frac{\eth}{2})}$$
$$= \cos\left(x+\acute{a}\frac{\eth}{2}\right) + isin\left(x+\acute{a}\frac{\eth}{2}\right) \qquad (3.2)$$

$$D^{\acute{a}}cosx - i\,D^{\acute{a}}si@x = D^{\acute{a}}[cosx - isinx]$$
$$= D^{\acute{a}}e^{-ix}$$
$$= (-i)^{\acute{a}}e^{-ix}$$
$$= \left(e^{\frac{-\eth i}{2}}\right)^{\acute{a}} e^{-ix}$$
$$= e^{-i\left(x+\acute{a}\frac{\eth}{2}\right)}$$
$$= \cos\left(x+\acute{a}\frac{\eth}{2}\right) - isin\left(x+\acute{a}\frac{\eth}{2}\right) \qquad (3.3)$$

Solving (3.2) and (3.3) we get

$$D^{\acute{a}}cosx = \cos\left(x+\acute{a}\frac{\eth}{2}\right), \quad \forall\acute{a}\in C$$
$$D^{\acute{a}}sinx = sin\left(x+\acute{a}\frac{\eth}{2}\right), \quad \forall\acute{a}\in C.$$

4. ELEMENTARY PROPERTIES OF FRACTIONAL DERIVATIVES:

For positive integer n, $D_t^n[kf(t)] = kD_t^n f(t)$, k is constant.

$D_t^n[f(t)+g(t)] = D_t^n f(t) + D_t^n g(\nearrow)$.

$D_t^n[f(t).g(t)] = \sum_{k=0}^{n}\binom{n}{k}D_t^{n-k}f(t)\,D_t^k g(t)$.

Proof (iii):

$$D_t^1[f(t).g(t)] = D_t^1 f(t) + D_t^1 g(t)$$
$$= \sum_{k=0}^{1}\binom{1}{k}D_t^{1-k}f(t)\,D_t^k g(t)$$
$$D_t^2[f(t).g(t)] = D[D_t^1 f(t) + D_t^1 g(t)]$$
$$= D_t^2 f(t)\,g(t) + 2D_t^1 f'(t)\,g(t) + f(t)D_t^2 g(t)]$$
$$= \sum_{k=0}^{2}\binom{2}{k}D_t^{2-k}f(t)\,D_t^k g(t)$$

Continuing in this way, we get result for any positive integer n.

This can be generalized for any $\acute{a}\in C$ as

$$D_t^{\acute{a}}[f(t).g(t)] = \sum_{k=0}^{\infty}\binom{\acute{a}}{k}D_t^{\acute{a}-k}f(t)\,D_t^k g(t) \qquad (4.1)$$

where

$$\binom{\acute{a}}{k} = \frac{\Gamma(\acute{a}+1)}{\Gamma(k+1)\Gamma(\acute{a}-k+1)}$$

and since $\acute{a}$ is not integer the limit of sum in (4.1) is ∞. Now if one of the functions in product is constant say $g(t) = C$ then from equation (4.1) we get

$$D_t^{\acute{a}}[f(t)C] = \sum_{k=0}^{\infty}\binom{\acute{a}}{k}D_t^{\acute{a}-k}f(t)\,D_t^k C$$
$$= D_t^{\acute{a}}f(t)\,C \qquad (4.2)$$

Moreover,

$$D_t^{\acute{a}}[f(t)+g(t)] = D_t^{\acute{a}}[t^0(f(t)+g(t))]$$
$$= \sum_{k=0}^{\infty}\binom{\acute{a}}{k}D_t^{\acute{a}-k}t^0 D_t^k[f(t)+g(t)]$$
$$= \sum_{k=0}^{\infty}\binom{\acute{a}}{k}D_t^{\acute{a}-k}t^0[D_t^k f(t) + D_t^k g(t)]$$
$$= \sum_{k=0}^{\infty}\binom{\acute{a}}{k}D_t^{\acute{a}-k}t^0 D_t^k f(t) + \sum_{k=0}^{\infty}\binom{\acute{a}}{k}D_t^{\acute{a}-k}t^0 D_t^k g(t)$$
$$= D_t^{\acute{a}}f(t) + D_t^{\acute{a}}g(t)$$

Therefore D^{α} is linear operator on set of functions whose $\acute{\alpha}^{t\square}$ derivative exists. (iv) From equations (4.2) and (4.3) the set *V* of all complex valued functions defined on [0, ∞) whose $\acute{\alpha}^{t\square}$ derivative exists is complex vector space with respect to addition of functions and scalar multiplication of function operations.

CONCLUSION

We study the gamma and beta functions of complex variable for the development of fractional calculus.
(ii) The elementary properties of fractional derivative are proved.

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