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VOLTERRA TYPE INTEGRAL INTEGRAL INEQUALITIES WITH SINGULAR KERNEL IN TWO VARIABLES

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ABSTRACT
In this paper we obtain some volterra-integral inequalities in two variables with singular kernel, which contains singularities at one or more points within the range of their variables.

KEY WORDS: Volterra integral inequality, singular karnal.
Subject Classification: - 45D, 45E.

INTRODUCTION
In this Paper, we study some volterra type integral inequalities in two variables with singular kernel and we also state and prove more general wendroff theorem [1, pp-109], where the right hand side is the solution of the corresponding integral equation. Instead we intend to investigate finite bound. We consider the interval [0, ∞) throughout.

Lemma 1.1 [1,pp-3] :- Let u(t) , f(t) , g(t) be continuous functions in J = [0, ∞) and g(t) be non-negative ; Let f(t) be nonnegative , non-decreasing in J and

if $u(t) \le f(t) + \int_0^t g(s)\, u(s)\, ds$, $t \in J$,

Then, $u(t) \le f(t) \exp\left(\int_0^t g(s)\, ds\right)$, $t \in J$

Preliminaries

Definition:- (Beta function)

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \quad , \quad \alpha > 0, \beta > 0.$$

$\therefore$ B(1- α , α) = B(α,β) = B (β , α). Where $\alpha + \beta$ =1. And B is beta function.

Definition: - (Gama function)

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy, \qquad x > 0. \qquad \ldots\ldots\ldots\ldots\ldots..(*)$$

On integration by parts $\Gamma(x+1) = \int_0^\infty y^{x} e^{-y} dy,$

$$= \left[-y^x e^{-y}\right]_0^\infty + x \int_0^\infty y^{x-1}. e^{-y} dy,$$

$$= x\,\Gamma(x) , \quad x > 0.$$

By using this recurrence relation when x= n, n being a positive integer ≥ 1. We have,

$$\Gamma(n+1) = n\,\Gamma(n) = n\,(n-1)\,\Gamma(n-1) = \ldots\ldots.. = n!\ \Gamma(1)$$

By (*) $\quad \Gamma(1) = \int_0^\infty e^{-y}\, dy = 1$

$\therefore \quad \Gamma(1) = 1$

$\therefore \quad \Gamma(n+1) = n!$ And $\Gamma(n) = (n-1)!$. If n is positive integer.

We state some properties of Beta function and Gamma function which are helpful to our results.

1. $\Gamma(n+1) = n!$
2. $\Gamma(1/2) = \sqrt{\pi}$
3. $\Gamma(n+1) = n\,\Gamma(n)$
4. $B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\quad\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, for $\alpha, \beta > 0$
5. $B(\alpha,1-\alpha) = \dfrac{\Gamma(\alpha)\quad\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)} = \Gamma(\alpha)\quad\Gamma(1-\alpha) = \dfrac{\pi}{\sin\alpha\pi}$

Lemma 1.2 :- For any x, y ϵ [0,∞) and $s \geq 0$, and $\alpha + \beta = 1$, $0 \leq \alpha < 1$. $0 \leq \beta < 1$.then

$$\int_s^x \frac{dy}{(x-y)^{1-\alpha}}\,\frac{}{(y-s)^{\alpha}} = \frac{\pi}{\sin\alpha\pi}$$

Proof:-

Put $\dfrac{x-y}{x-s} = u$

$\Rightarrow$ y = x – u (x-s)

$\therefore$ dy = -(x -s) du

$\therefore$ limits; y = x, u = 0

y = s, u = 1

$$\therefore \quad I = \int_{u=1}^{u=0} \frac{}{u^{1-\alpha}}\,\frac{(x-s)\quad du}{(x-s)^{1-\alpha}\quad u^{\alpha}\quad (x-s)^{\alpha}} =$$

$$= \int_0^1 u^{\alpha-1}(1-u)^{\alpha}\quad du = \int_0^1 u^{\alpha-1}(1-u)^{(1-\alpha)-1}\quad du$$

$$= B(\alpha,1-\alpha)$$

Using property [5], we obtain

$$= \frac{\Gamma(\alpha)\quad\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)} = \Gamma(\alpha)\quad\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\pi}$$

Corrollary 1.2:- For any x, y ϵ [0,∞) and $s \geq 0$, and $\alpha + \beta = 1$, $\alpha, \beta > 0$ 1. $\alpha = \beta = ½$

$$\text{Then} \quad \int_s^x \frac{dy}{(x-y)^{1-\alpha}}\,\frac{}{(y-s)^{\alpha}} = \int_s^x \frac{dy}{\sqrt{x-y}\quad\sqrt{y-s}} = \pi$$

Proof: - In lemma 1.2, $\alpha = \beta = ½$

$$B(\alpha,1-\alpha) = B(\tfrac{1}{2},\tfrac{1}{2}) = \frac{\sqrt{\frac{1}{2}}\quad\sqrt{\frac{1}{2}}}{\sqrt{(\frac{1}{2}+1)-\frac{1}{2}}} = \Gamma(1/2)\quad\Gamma(1/2)$$

$\therefore$ Using property [2]

$$= \sqrt{\pi}\quad\sqrt{\pi} = \pi$$

Lemma 1.3 [3] :- Let u (t) be non negative continuous function on $[0, \infty)$ and assume that,

$$u(t) \le a + b\int_0^t \frac{u(s)\,ds}{\sqrt{t-s}}$$ then for some constants $a, b \ge 0$ and for every $t \ge 0$,

$$u(t) \le a\,[\,1 + 2b\sqrt{t}\,]\exp(\pi b^2 t)$$

Proof: - Substitute of the equation into itself and evaluation of the integration gives then

$$u(t) \le a + b\int_0^t \frac{a}{\sqrt{t-s}}ds + b^2\int_0^t\int_0^s \frac{1}{\sqrt{s-\tau}}\frac{1}{\sqrt{t-s}}\,u(\tau)\,d\tau\,ds$$

then, $$u(t) \le a\,[\,1 + 2b\sqrt{t}\,] + \exp b^2\int_0^t u(\tau)d\tau$$

Application of the Lemma 1.1 and corollary 1.2 leads to the desired result.

We now consider integral Inequalities of functions of two variables

2. Volterra Integral Inequalities in two variables

Theorem 2.1 [1, pp-109] Let u(x,y) , a(x,y) , k(x,y) be non-negative continuous function for $x \ge x_0$, $y \ge y_0$ and let a(x,y) be non-decreasing in each of the variables for $x \ge x_0$, $y \ge y_0$. Suppose that,

$$u(x,y) \le a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\,u(s,t)\,ds\,dt \quad , \quad x \ge x_0 \ , y \ge y_0 \ . \qquad (2.1)$$

$$\text{then, } u(x,y) \le a(x,y)\exp\Big(\int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\,ds\,dt\Big) \quad , \quad x \ge x_0 \ , y \ge y_0 \ . \qquad (2.2)$$

Proof:- Let $X \ge x_0$, $Y \ge y_0$ be fixed then for $x_0 \le x \le X$, $y_0 \le y \le Y$ setting right side of (2.1) is $V(x,y)$. The function $V(x,y)$ is non-decreasing in each variable

x , y and $V(x_0,y) = a(x,y)$.

$$\frac{\partial}{\partial x}V(x,y) = \int_{y_0}^{y} k(x,t)\,u(x,t)\,dt \le \int_{y_0}^{y} k(x,t)\,V(x,t)\,dt$$

Since $u(x,y) \le V(x,t) \le V(x,y)$ by theorem 1.3 [1]

$$V(x,y) \le p(x,y)\exp\Big(\int_{x_0}^{x}\Big[\int_{y_0}^{y} k(s,t)\,dt\Big]\Big)ds \ , \qquad x_0 \le x \le X \ , \ y_0 \le y \le Y$$

Putting $x = X$, $y = Y$ and we arrive at (2.1)

Result: - 1. Lemma 1.3 [2] Baigioni's inequality, we have proved

$$\int_\tau^t \frac{ds}{\sqrt{t-s}\,\sqrt{s-t}} = \pi$$

As an application of corollary 1.2, we obtain desired result.

Result: - 2. Here we use some term for double integration and obtain an interesting identity

Here we use some term for double integral and obtain an interesting identity

$$\int_\xi^x\int_\eta^y \frac{1}{\sqrt{s-\xi}}\frac{ds}{\sqrt{x-s}}\frac{dt}{\sqrt{y-t}}\frac{1}{\sqrt{t-\eta}} = \pi^2$$

As an application of corollary 1.2 (twice), we obtain desired result.

Lemma 2.2 :- For any x, y ∈[0,∞) and s , t ≥ 0. $\alpha + \beta = 1$, $\gamma + \delta = 1$, and $\alpha = \beta = \gamma = \delta = 1/2$.

$$\int_{\xi}^{x}\int_{t}^{y}\frac{ds}{\sqrt{s-\xi}}\frac{ds}{\sqrt{x-s}}\frac{dt}{\sqrt{y-t}}\frac{}{\sqrt{t-\eta}} = \pi^2$$

Proof: - Put $\frac{x-s}{x-\xi} = u$

$\Rightarrow$ $s = x - u(x-\xi)$ and $\frac{y-t}{y-\eta} = v$ $\Rightarrow$ $t = y - (y-\eta)v$

When $s = x$, $u = 0$ $\quad$ $t = y$, $v = 0$ and

$s = \xi$, $u = 1$ $\quad$ $t = \eta$, $v = 1$

As an application of property [5] (twice), we obtain desired result.

Now we obtain new version of Biagionis inequality [3] in two variables.

Theorem 2.3 Let u (x,y) be non-negative continuous function for $x \geq x_0$, $y \geq y_0$ and p,q be non-zero constants. Suppose that,

$$u(x,y) \leq p + q \int_0^x\int_0^y \frac{u(s,t)}{\sqrt{x-s}\ \sqrt{y-t}}\, ds\, dt \qquad (2.3)$$

then $u(x,y) \leq p\ [\,1+4q\sqrt{x}\ \sqrt{y}\,]\exp(\pi^2 q^2 x\, y)$ (2.4)

proof:- Substitute into itself we obtain,

$$u(x,y) \leq p + p\,q \int_0^x\int_0^y \frac{ds\ dt}{\sqrt{x-s}\ \sqrt{y-t}} + q^2\int_0^x\int_0^y \Big(\int_{\xi}^{x}\int_{\eta}^{y}\frac{ds\ dt}{\sqrt{x-s}}\frac{}{\sqrt{s-\xi}\ \sqrt{y-t}\ \sqrt{t-\eta}}\Big)\, u(\xi,\eta)\, d\xi\, d\eta$$

Using corollary 1.2, lemma 1.1 and lemma 1.3 we get required result. (2.4)

Theorem 2.4- Suppose that p(x,y), u(x,y) be non-negative continuous functions for $x \geq 0$, $y \geq 0$ and

$$u(x,y) \leq p(x,y) + q \int_0^x\int_0^y \frac{u(s,t)}{\sqrt{x-s}\ \sqrt{y-t}}\, ds\, dt\ .\quad x \geq 0\ ,\ y \geq 0 \qquad (2.5)$$

then $u(x,y) \leq p(x,y)\ [\,1+4q\sqrt{x}\ \sqrt{y}\,]\exp(\pi^2 q^2 x\, y)$ (2.6)

Proof :- Consider p (x,y) ≥ 0 for x > 0 , y > 0

$$\therefore \frac{u(x,y)}{p(x,y)} \leq 1 + q\int_0^x\int_0^y \frac{u(s,t)\,ds\,dt}{p(s,t)\sqrt{x-s}\ \sqrt{y-t}}$$

Let, $R = \frac{u(s,t)}{p(s,t)}$, then above equation becomes as

$$R(x,y) \leq 1 + \int_0^x\int_0^y \frac{R(s,t)}{\sqrt{x-s}\ \sqrt{y-t}}\, ds\, dt$$

By above theorem 2.1 and lemma 2.2

We get, $R(x,y) \leq [\,1+4q\sqrt{x}\ \sqrt{y}\,]\exp(\pi^2 q^2 x\, y)$. This gives the result (2.6).

Let p(x,y) = 0 and ϵ_1 , ϵ_2 > 0

$$u(x,y) \leq (\epsilon_1,\epsilon_2) + b\int_0^x\int_0^y \frac{u(s,t)}{\sqrt{x-s}\ \sqrt{y-t}}\, ds\, dt$$

then, $u(x,y) \leq (\epsilon_1,\epsilon_2)\ [\,1+4q\sqrt{x}\ \sqrt{y}\,]\exp(\pi^2 q^2 x\, y)$.

Letting $\epsilon 1, \epsilon 2 \to 0$ we find that, thus u(x,y) =0.
Now we prove a more general inequality of theorem (2.3)

Theorem 2.5- Let u(x,y) , p(x,y) be continuous functions on $J \times J \subset R_+ \times R_+$,
Let H (x,y) , K (x,y) be square integrable and continuous on $J \chi J$ and
Suppose $0 < \alpha < 1$, $0 < \gamma < 1$.

$$u(x,y) \le (x-s)^{\alpha}(y-t)^{\gamma}\, p(x,y) + \int_0^x \int_0^y \frac{H(x,s)\quad K(y,t)\quad u(s,t)\,ds\,dt}{(x-s)^{\alpha}\quad (y-t)^{\gamma}} \tag{2.7}$$

then $u(x,y) \le V(x,y)$, (x,y) ε J ×J (2.8)

where V (x,y) is the solution of the equation

$$V(x,y) = (x-s)^{\alpha}(y-t)^{\gamma}\, p(x,y) + \int_0^x \int_0^y H(x,s)\, Y(y,t)\, P(s,t)\, ds\, dt$$

$$+ \frac{\pi^2}{\sin(\pi\alpha)\quad \sin(\alpha\gamma)} \int_0^x \int_0^y H(x,s)\, K(y,t)\, V(s,t)\, ds dt \tag{2.9}$$

Proof: - Substitute u(x,y) in to itself we obtain

$$u(x,y) \le (x-s)^{\alpha}(y-t)^{\gamma}\, p(x,y) + \int_0^x \int_0^y H(x,s)\, K(y,t)\, P(s,t)\, ds\, dt$$

$$+\int_0^x \int_0^y \left[\int \int^{xy} \frac{ds \quad dt}{s\,t\,(x-s)^{\alpha}\quad (s-\tau)^{\beta}\quad (y-t)^{\gamma}\quad (t-\tau')^{\delta}} \right] H^2(s,\tau)\, K^2(t,\zeta)\, d\tau\ d\zeta.$$

$$\text{Now } \int \int^{xy} \frac{ds \quad dt}{s\,t\,(x-s)^{\alpha}\quad (s-\tau)^{\beta}\quad (y-t)^{\gamma}\quad (t-\tau')^{\delta}} = \frac{\pi^2}{\sin(\pi\alpha)\quad \sin(\alpha\gamma)}, \text{ where } \alpha+\beta=1;\ \gamma+\delta=1. \tag{2.10}$$

Using above value in (2.10), we get required result (2.8); Note that H(x,s) , K(y,t) are continuous and square integrable. Hence middle factor of equation (2.9)can be integrated , if V (x,y) is the solution of (2.9) .
Thus we have $u(x,y) \le V(x,y)$, (x,y) ε $J \chi J$.

Remark 2.5.1- In theorem 2.5 if $\alpha = 0$, $\gamma = 0$ and H (x,s) = 1 then it reduces to the theorem 2.1 [1,pp-109]
The following corollary is particular case of theorem 2.5 It is similar to Biagoni's theorem in two variables and which is more general to it.

Corollary 2.5.2- Let $\alpha = ½$, $\gamma = ½$ In above theorem and H (x,y), K (x,y) are continuous and square integrable and p(x,y) be continuous as in theorem 2.5

$$u(x,y) \le (x-s)^{½}(y-t)^{½}\, p(x,y) + \int_0^x \int_0^y H(x,s)\, K(y,t)\, P(s,t)\, ds\, dt + \pi^2 \int_0^x \int_0^y H(x,s)\, K(y,t)\, V(s,t)\, ds\, dt$$

The proof is on same line of theorem (2.5)

Corollary 2.5.3:- If In theorem 2.5 p (x,y) = p(x)+ q(y) where p(x) and q(y) are non-decreasing functions for $x \ge 0$, $y \ge 0$ then

$$u(x,y) \le \left| p(x)+q(y) \right| (\exp \int_0^x \int_0^y K(s,t)\, ds\, dt\ ).$$

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