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LENGTH OF OPEN SETS IN R VIA RIEMANN INTEGRAL AND ITS APPLICATIONS

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ABSTRACT

The aim of this paper is to define length of bounded open set in R by using Riemann integral of its characteristic function and use it to give alternate simple proofs of known results on length of open sets.

KEY WORDS: Characteristic function, Closed set, Length of set, Open set, Riemann integral,

INTRODUCTION

Usually length of open set G of closed and bounded interval $[a, b]$ is defined by using length of open intervals. Lengths of open and closed sets are used to define outer and inner measure of sets respectively. Via these measures we can define measurable sets which are used in Lebesgue integral of real function defined on$[a, b]$. We organize this paper as follows:

Section (2) is devoted for some definitions required to prove main result. In section (3) we prove property that length of interval is Riemann integral of its characteristic function. Section (4) is devoted for alternate proofs of main results.

Definitions:

In this section we discuss definitions of intervals, open set and closed set. We also define set of measure zero, characteristic function, and property hold a. e.

i. Length of interval: Let I be an interval of type $[a, b], (a, b), [a,, b)$ or$(a, b]$. Then length of I is $|I\ | = b - a.$
ii. Set of measure zero: A subset E of R (Set of real numbers) is of measure zero if for given $\varepsilon > 0$ there exist finite or countable number of pair wise disjoint open intervals $I_1, I_2, I_3, \ldots$ such that $E \subseteq \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \varepsilon.$
iii. Property hold almost everywhere (a. e.): Let $P(x)$ is property related with real variable x. Then we say that $P(x)$ hold a. e. if the set $\{x \in R \ / \ P(x)\ doesn't\ hold\}$ is of measure zero.
iv. Open set: A subset G of an interval $[a, b]$ is said to be open set if for given $x \in G$, there exist $\delta > 0$ such that $(x - \delta, x + \delta) \bigcap [a, b] \subseteq G.$
v. Closed set: A subset F of interval $[a, b]$ is closed if complement of F that is $F^c = [a, b] - F$ is open in$[a, b]$.
vi. Characteristic function: Let G be a subset of nonempty set X. Then a function $\chi_G : X \to R$ is called as characteristic function if $\chi_G(x) = 1$ for $x \in G$ and $\chi_G(x) = 0$ for $x \in G^c = X - G$.
vii. Length of open set: Let G be an open subset of a closed and bounded interval $[a, b]$. Then $G = \cup\ I_n$ where $I_1, I_2, I_3, \ldots$ are finite or countable numbers of pair wise disjoint open intervals in$[a, b]$. Then length of open set is G is denoted by $|G|$ and it is defined as$|G| = \sum |I_n|$.
viii. Length of closed set: Let F be a closed subset of a closed and bounded interval $[a, b]$. Then $F^c = [a, b] - F$ is an open subset of$[a, b]$ and length of F is $|F| = b - a - |F^c|$.

Properties of Lengths

Theorem 3.1: Let I be subinterval of $[a, b]$ and $\chi_I : [a, b] \to R$ be the characteristic function on I. Then χ_I is Riemann integrable function and

$$\int_a^b \chi_I = |I|.$$

Proof: By definition of characteristic function,

$$\chi_I\ (x) = 1\ if\ x \in I$$
$$= 0\ if\ x \in\ I^c = [a, b] - I$$

Now range of χ_I is subset of {0, 1} and hence χ_I is bounded function. Moreover χ_I is discontinuous at almost two points which are extreme points of an interval I and finite set is of measure zero implies that χ_I is continuous a. e. Therefore χ_I is Riemann integrable function. Now, if $I = [c, d] \subseteq [a, b]$ then

$$\int_a^b \chi_I = \int_a^c \chi_I + \int_c^d \chi_I + \int_d^b \chi_I$$
$$= \int_a^c 0 + \int_c^d 1 + \int_d^b 0$$
$$= d - c = |I|\ .$$

Thus the result is proved.

Theorem 3.2: Let G be an open subset of the closed and bounded interval $[a, b]$ then the characteristic function χ_G of G is Riemann integrable in $[a, b]$ and

$$\int_a^b \chi_G = |G|$$

Proof: Since G is an open subset of $[a,b]$, therefore it is expressed as $G = \cup I_n$ where $I_1, I_2, I_3, ...$ are finite or countable numbers of pair wise disjoint open intervals in $[a,b]$.
By definition of characteristic function,

$$\chi_G\ (x) = 1\ if\ x \in G$$
$$= 0\ if\ x \in\ G^c = [a,b] - G$$

Now range of χ_G is subset of {0, 1} and hence χ_G is bounded function. Moreover χ_G is discontinuous at almost extreme points of an intervals $I_n's$ and this set is finite, countable or uncountable of measure zero implies that χ_G is continuous a. e. Therefore χ_G is Riemann integrable function.
Now,

$$\chi_G\ (x) = \sum \chi_{I_n}(x);\ \ x \in [a,b].$$

Therefore,

$$\int_a^b \chi_G\ = \int_a^b \sum \chi_{I_n}\ = \sum \int_a^b \chi_{I_n} = \sum |I_n| = |G|.$$

Thus result is proved.

Theorem 3.3: Let G be an open subset of the closed and bounded interval $[a,b]$ with $|G| = 0$ then $G = \emptyset$ (empty set).
Proof: Suppose G be nonempty open subset of the closed and bounded interval $[a,b]$ then for $c \in G, \exists\ \delta > 0$ such that $I = (c-\delta, c+\delta) \cap\ [a,b] \subseteq G \subseteq [a,b]$.
Now $I\ and\ G$ are open subsets of $[a,b]$ implies that $\chi_I\ and\ \chi_G$ are Riemann integrable functions in $[a,b]$. Moreover,

$$\chi_I(x) \leq\ \ \chi_G(x);\ \forall x \in [a,b]\ .$$

Therefore,

$$\int_a^b \chi_I\ \leq \int_a^b \chi_G$$
$$\therefore 0 < |I| = 2\delta \leq |G|$$

This is contradiction to $|G| = 0$ and hence G must be empty set.

ALTERNATE PROOFS OF MAIN RESULTS

Theorem 4.1: Let G_1, G_2 be the open subsets of a closed and bounded interval $[a,b]$ then,

$$|G_1| + |G_2| = |G_1\ \cup G_2| + |G_1\ \cap G_2|.$$

Proof: $\because\ G_1, G_2$ are open subsets a closed and bounded interval $[a,b]$.
$\therefore\ G_1\ \cup G_2,\ \ G_1\ \cap G_2$ are also open subsets of $[a,b]$.
Since characteristic function of any open set G of $[a,b]$ is Riemann integrable function implies that $\chi_{G_1}, \chi_{G_2}, \chi_{G_1 \cup G_2}\ and\ \chi_{G_1 \cap G_2}$ are also Riemann integrable functions in $[a,b]$.
Moreover,

$$\chi_{G_1}(x) + \chi_{G_2}(x) = \chi_{G_1 \cup G_2}(x) +\ \chi_{G_1 \cap G_2}(x), \forall x \in [a,b].$$

Therefore,

$$\int_a^b \chi_{G_1} + \chi_{G_2} = \int_a^b \chi_{G_1 \cup G_2} +\ \chi_{G_1 \cap G_2}$$
$$\therefore \int_a^b \chi_{G_1} + \int_a^b \chi_{G_2} = \int_a^b \chi_{G_1 \cup G_2} + \int_a^b \chi_{G_1 \cap G_2}$$
$$\therefore |G_1| + |G_2| = |G_1\ \cup G_2| + |G_1\ \cap G_2|.$$

Thus the result is proved.

Theorem 4.2: Let G_1, G_2, G_3 .... be the open subsets of a closed and bounded interval $[a,b]$ then,

$$|\cup\ G_n| \leq \sum |G_n|\ .$$

Proof: Proof by induction,
If G_1, G_2 are open subsets of $[a,b]$ then by theorem 4.2,

$$|G_1| + |G_2| = |G_1\ \cup G_2| + |G_1\ \cap G_2|.$$
$$\therefore |G_1\ \cup G_2| \leq |G_1| + |G_2|\ \text{ as }\ |G_1\ \cap G_2| \geq 0.$$

Thus result is true for $n = 2$.
Suppose result is true for positive integer k that is

$$\left|\cup_{n=1}^k G_n\right| \leq \sum_{n=1}^k |G_n|\ .$$

Then,

$$\left|\bigcup_{n=1}^{k+1} G_n\right| = \left|\left(\bigcup_{n=1}^{k} G_n\right) \bigcup G_{k+1}\right|$$
$$\leq \left|\cup_{n=1}^k G_n\right| + |G_{k+1}|\ \ \text{(By theorem 4.1)}$$
$$\leq\ \sum_{n=1}^k |G_n| + |G_{k+1}| = \sum_{n=1}^{k+1} |G_n|\ \ \text{(By induction hypothesis)}$$

Thus result is true for $k+1$.

Therefore by induction result is true for any positive integer *k.*

Therefore,

$$\left|\bigcup_{n=1}^{k} G_n\right| \leq \sum_{n=1}^{\infty} |G_n|$$

Therefore as $k \to \infty$, we get

$$\left|\bigcup_{n=1}^{\infty} G_n\right| \leq \sum_{n=1}^{\infty} |G_n|$$

Thus result is proved.

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